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ON A MECHANICAL ANALOGY IN THE IDEAL PLASTICITY THEORY

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The Cauchy problem of propagation of plastic state zones in a boundless medium from the boundary of a convex surface, along which normal pressure and shear forces act, is considered. In the case of complete plasticity, the Tresca system of quasi-static equations of ideal plasticity, which describes the stress-strain state of the medium, is known to be hyperbolic and to be similar to a system that describes a steady-state flow of an ideal incompressible fluid. This system is numerically solved with the use of a difference scheme applied for hyperbolic systems of conservation laws. Results of numerical calculations are presented.

Key words: *Tresca ideal plasticity, complete plasticity, support function of a contour, equidistant surface, hyperbolic system of conservation laws.*

In the case of complete plasticity, the quasi-static equations of the Tresca ideal plasticity for determining stresses have the form [1]

$$\frac{\partial \boldsymbol{t}_1}{\partial x_1} + \frac{\partial \boldsymbol{t}_2}{\partial x_2} + \frac{\partial \boldsymbol{t}_3}{\partial x_3} = 0; \qquad (1)$$

$$\boldsymbol{t}_1 = (\sigma_{11}, \sigma_{12}, \sigma_{13})^{\rm t}, \qquad \boldsymbol{t}_2 = (\sigma_{21}, \sigma_{22}, \sigma_{23})^{\rm t}, \qquad \boldsymbol{t}_3 = (\sigma_{31}, \sigma_{32}, \sigma_{33})^{\rm t}; \qquad (2)$$

$$\begin{gathered} \sigma_{ij} = \sigma\delta_{ij} + 2k\varepsilon(n_i n_j - \delta_{ij}/3), \qquad \sigma = \sigma_{ij}\delta_{ij}/3, \qquad \varepsilon = \pm 1, \\ n_1^2 + n_2^2 + n_3^2 = 1, \end{gathered} \qquad (3)$$

where σ_{ij} are the components of the symmetric stress tensor in the Cartesian coordinate system (x_1, x_2, x_3) with the basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, σ is the mean stress; δ_{ij} is the Kronecker delta, summation is performed over repeated indices from 1 to 3, k is the yield stress at pure shear, and $\boldsymbol{n} = n_i\boldsymbol{e}_i$ is the unit eigenvector corresponding to the principal stress

$$\sigma_1 = \sigma + 4k\varepsilon/3, \qquad (4)$$

and two other principal stresses coincide with each other:

$$\sigma_2 = \sigma_3 = \sigma - 2k\varepsilon/3. \qquad (5)$$

The stress tensor defined by equalities (2) with allowance for (3) satisfies the Tresca plasticity condition

$$\max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) = 2k.$$

The functions σ, n_1, n_2, and n_3 depending on x_1, x_2, and x_3 are determined from the first-order system of equations obtained by substituting Eqs. (2) and (3) into (1). This system is hyperbolic and has the form

$$n_1 \frac{\partial n_1}{\partial x_1} + n_2 \frac{\partial n_1}{\partial x_2} + n_3 \frac{\partial n_1}{\partial x_3} + n_1\theta = -\frac{\partial p}{\partial x_1},$$

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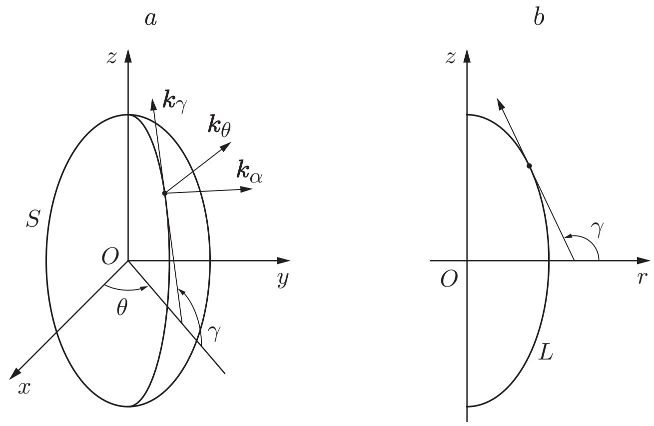

Fig. 1. Orthogonal curvilinear coordinate system (α, θ, γ): (a) coordinate surface S ($\alpha = 0$) and basis vectors $\boldsymbol{k}_\alpha$, $\boldsymbol{k}_\theta$, and $\boldsymbol{k}_\gamma$; (b) generatrix L of the coordinate surface S.

$$n_1 \frac{\partial n_2}{\partial x_1} + n_2 \frac{\partial n_2}{\partial x_2} + n_3 \frac{\partial n_2}{\partial x_3} + n_2 \theta = -\frac{\partial p}{\partial x_2},$$

$$n_1 \frac{\partial n_3}{\partial x_1} + n_2 \frac{\partial n_3}{\partial x_2} + n_3 \frac{\partial n_3}{\partial x_3} + n_3 \theta = -\frac{\partial p}{\partial x_3}, \tag{6}$$

$$n_1^2 + n_2^2 + n_3^2 = 1, \qquad \theta = \frac{\partial n_1}{\partial x_1} + \frac{\partial n_2}{\partial x_2} + \frac{\partial n_3}{\partial x_3}, \qquad p = \frac{\sigma}{2k\varepsilon} - \frac{1}{3}.$$

The characteristic relations for system (6) were derived in [2].

System (6), which is similar to a system that describes a steady-state flow of an ideal incompressible fluid [3, p. 73], can be written as [4, p. 105]

$$2(\boldsymbol{\omega} \times \boldsymbol{n}) + \theta \boldsymbol{n} = -\nabla p, \qquad n_1^2 + n_2^2 + n_3^2 = 1, \qquad \boldsymbol{\omega} = (1/2)\nabla \times \boldsymbol{n}. \tag{7}$$

By analogy to [4], we can consider the following cases.

Case 1. The unit eigenvector is presented as

$$\boldsymbol{n} = \mu \nabla f, \tag{8}$$

where $f = f(x_1, x_2, x_3)$ is a certain function and $\mu = |\nabla f|^{-1}$.

Case 2. The unit eigenvector is presented in the form

$$\boldsymbol{n} = \lambda \boldsymbol{\omega}, \tag{9}$$

where $\lambda = |\boldsymbol{\omega}|^{-1}$. In this case, Eq. (7) becomes substantially simplified:

$$\theta \boldsymbol{n} = -\nabla p.$$

Case 3. The following equality is valid:

$$\boldsymbol{\omega} \cdot \boldsymbol{n} = 0. \tag{10}$$

To determine the functions $p = p(x_1, x_2, x_3)$ and $n_i = n_i(x_1, x_2, x_3)$, we need compatibility of systems (6) and (8) in case 1, (6) and (9) in case 2, and (6) and (10) in case 3. These variants were considered in [5, 6].

To solve the problem of propagation of plastic state zones from the boundary of a convex surface subjected to normal pressure and shear forces, we use numerical algorithms applied in hydromechanics [7–9].

Let S be a rather smooth closed convex surface, with the origin of the coordinate system (x, y, z) lying inside S (Fig. 1a). The surface S is formed by rotation of a convex curve L (Fig. 1b) located in the plane (r, z), where $r = (x^2 + y^2)^{1/2}$; the curve L crosses the z axis at right angles, and the curvature radius ρ of the curve L at an arbitrary point of the curve is equal to or greater than $\rho_* > 0$.

We write the equations of the curve L in parametric form:

$$r(\gamma) = \frac{dF(\gamma)}{d\gamma}\cos\gamma + F(\gamma)\sin\gamma, \qquad z(\gamma) = \frac{dF(\gamma)}{d\gamma}\sin\gamma - F(\gamma)\cos\gamma. \tag{11}$$

Here γ is the angle between the tangent line and the r axis and $F(\gamma)$ is the support function of the contour L. Obviously, $r(\gamma) \geqslant 0$ for $0 \leqslant \gamma \leqslant \pi$. The curvature radius of the curve L is calculated by the formula

$$\rho = \rho(\gamma) = \frac{1}{\cos\gamma}\frac{dr(\gamma)}{d\gamma} = F(\gamma) + \frac{d^2F(\gamma)}{d\gamma^2} \geqslant \rho_* > 0. \tag{12}$$

The equations for the surface S are written as

$$\begin{aligned} x_S(\theta,\gamma) &= \Big(\frac{dF(\gamma)}{d\gamma}\cos\gamma + F(\gamma)\sin\gamma\Big)\cos\theta, \\ y_S(\theta,\gamma) &= \Big(\frac{dF(\gamma)}{d\gamma}\cos\gamma + F(\gamma)\sin\gamma\Big)\sin\theta, \\ z_S(\theta,\gamma) &= \frac{dF(\gamma)}{d\gamma}\sin\gamma - F(\gamma)\cos\gamma. \end{aligned} \tag{13}$$

We introduce an orthogonal curvilinear coordinate system (α, θ, γ):

$$\begin{aligned} x(\alpha,\theta,\gamma) &= \Big(\frac{dF(\gamma)}{d\gamma}\cos\gamma + \Big(F(\gamma)+\alpha\Big)\sin\gamma\Big)\cos\theta, \\ y(\alpha,\theta,\gamma) &= \Big(\frac{dF(\gamma)}{d\gamma}\cos\gamma + \Big(F(\gamma)+\alpha\Big)\sin\gamma\Big)\sin\theta, \\ z(\alpha,\theta,\gamma) &= \frac{dF(\gamma)}{d\gamma}\sin\gamma - \Big(F(\gamma)+\alpha\Big)\cos\gamma, \\ \alpha \geqslant 0, \qquad & 0 \leqslant \gamma \leqslant \pi, \qquad 0 \leqslant \theta \leqslant 2\pi. \end{aligned} \tag{14}$$

The Jacobian of coordinate transformation $J(\alpha, \theta, \gamma)$ is non-negative:

$$J(\alpha,\theta,\gamma) = \frac{D(x,y,z)}{D(\alpha,\theta,\gamma)} = \Big(\rho(\gamma)+\alpha\Big)\Big(r(\gamma)+\alpha\sin\gamma\Big) \geqslant 0.$$

It follows from Eq. (14) that the surface $\alpha = \text{const}$ is an equidistant surface with respect to the surface S. The expressions for the unit vectors of the coordinate lines (see Fig. 1a) have the form

$$\begin{aligned} &\boldsymbol{k}_\alpha = (\sin\gamma\cos\theta, \sin\gamma\sin\theta, -\cos\gamma)^{\mathrm{t}}, \\ &\boldsymbol{k}_\theta = (-\sin\theta, \cos\theta, 0)^{\mathrm{t}}, \qquad \boldsymbol{k}_\gamma = (\cos\gamma\cos\theta, \cos\gamma\sin\theta, \sin\gamma)^{\mathrm{t}}. \end{aligned} \tag{15}$$

We write the equilibrium equations in the orthogonal coordinate system (α, θ, γ). The stresses $\sigma_{\alpha\alpha}$, $\sigma_{\alpha\theta}$, $\sigma_{\alpha\gamma}$, $\sigma_{\theta\theta}$, $\sigma_{\theta\gamma}$, and $\sigma_{\gamma\gamma}$ satisfy the equations

$$\frac{\partial \boldsymbol{t}_\alpha}{\partial\alpha} + \frac{\partial \boldsymbol{t}_\theta}{\partial\theta} + \frac{\partial \boldsymbol{t}_\gamma}{\partial\gamma} = 0, \tag{16}$$

where

$$\begin{aligned} &\boldsymbol{t}_\alpha = H_\theta H_\gamma(\sigma_{\alpha\alpha}\boldsymbol{k}_\alpha + \sigma_{\alpha\theta}\boldsymbol{k}_\theta + \sigma_{\alpha\gamma}\boldsymbol{k}_\gamma), \\ &\boldsymbol{t}_\theta = H_\alpha H_\gamma(\sigma_{\alpha\theta}\boldsymbol{k}_\alpha + \sigma_{\theta\theta}\boldsymbol{k}_\theta + \sigma_{\theta\gamma}\boldsymbol{k}_\gamma), \qquad \boldsymbol{t}_\gamma = H_\alpha H_\theta(\sigma_{\alpha\gamma}\boldsymbol{k}_\alpha + \sigma_{\theta\gamma}\boldsymbol{k}_\theta + \sigma_{\gamma\gamma}\boldsymbol{k}_\gamma); \end{aligned} \tag{17}$$

$$H_\alpha = 1, \qquad H_\theta = r(\gamma) + \alpha\sin\gamma, \qquad H_\gamma = \rho(\gamma) + \alpha. \tag{18}$$

Eliminating basis vectors (15) from system (16) with allowance for Eq. (17), we write system (16) in the form of a system of conservation laws

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial\alpha} + \frac{\partial \bar{\boldsymbol{v}}}{\partial\theta} + \frac{\partial \bar{\boldsymbol{w}}}{\partial\gamma} = \boldsymbol{f}, \tag{19}$$

where

$$\bar{\boldsymbol{u}} = H_\theta H_\gamma \boldsymbol{u}, \qquad \bar{\boldsymbol{v}} = H_\gamma \boldsymbol{v}, \qquad \bar{\boldsymbol{w}} = H_\theta \boldsymbol{w}; \tag{20}$$

$$\boldsymbol{u} = (\sigma_{\alpha\alpha}, \sigma_{\alpha\theta}, \sigma_{\alpha\gamma})^{\mathrm{t}}, \qquad \boldsymbol{v} = (\sigma_{\alpha\theta}, \sigma_{\theta\theta}, \sigma_{\theta\gamma})^{\mathrm{t}}, \qquad \boldsymbol{w} = (\sigma_{\alpha\gamma}, \sigma_{\theta\gamma}, \sigma_{\gamma\gamma})^{\mathrm{t}}; \tag{21}$$

$$\boldsymbol{f} = \begin{pmatrix} H_\gamma \sigma_{\theta\theta} \sin\gamma + H_\theta \sigma_{\gamma\gamma} \\ -H_\gamma(\sigma_{\theta\gamma}\cos\gamma + \sigma_{\alpha\theta}\sin\gamma) \\ (\partial H_\theta/\partial\gamma)\sigma_{\theta\theta} - H_\theta \sigma_{\alpha\gamma} \end{pmatrix};$$

$$\sigma_{\alpha\alpha} = p + n_\alpha^2, \qquad \sigma_{\alpha\theta} = n_\alpha n_\theta, \qquad \sigma_{\alpha\gamma} = n_\alpha n_\gamma; \tag{22}$$

$$\sigma_{\theta\theta} = p + n_\theta^2, \qquad \sigma_{\gamma\gamma} = p + n_\gamma^2, \qquad \sigma_{\theta\gamma} = n_\theta n_\gamma. \tag{23}$$

In expressions (22) and (23), the stresses $\sigma_{\alpha\alpha}$, $\sigma_{\alpha\theta}$, $\sigma_{\alpha\gamma}$, $\sigma_{\theta\theta}$, $\sigma_{\gamma\gamma}$, and $\sigma_{\theta\gamma}$ are the components of the stress σ_{ij} (3) in the orthogonal curvilinear coordinate system (α, θ, γ) divided by $2k\varepsilon$; $p = \sigma/(2k\varepsilon) - 1/3$. The quantities n_α, n_θ, and n_γ are the components of the unit eigenvector $\boldsymbol{n}$ corresponding to the principal stress (4) in the coordinate system (α, θ, γ):

$$\boldsymbol{n} = n_\alpha \boldsymbol{k}_\alpha + n_\theta \boldsymbol{k}_\theta + n_\gamma \boldsymbol{k}_\gamma, \qquad n_\alpha^2 + n_\theta^2 + n_\gamma^2 = 1. \tag{24}$$

The vector equation (19) and relation (24) form a closed system for determining the values of p, n_α, n_θ, and n_γ as functions of α, θ, and γ.

For system (19), (24), we pose the following Cauchy problem. The initial values of the functions p, n_α, n_θ, and n_γ are set on the surface S defined by formulas (13), i.e., at $\alpha = 0$, $0 \leqslant \theta \leqslant 2\pi$, and $0 \leqslant \gamma \leqslant \pi$. We have to find the values of these functions for $\alpha > 0$. In such a formulation of the Cauchy problem, the quantity α is an independent evolutionary variable, and system (19), (24) is α-hyperbolic, at least near the surface S [7, 8]. For the numerical solution of this system, therefore, we can use standard difference schemes applied to hyperbolic systems of conservation laws [7, 9].

In a rectangular domain

$$\Pi = \{\theta, \gamma\colon 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \gamma \leqslant \pi\}$$

we introduce a uniform difference grid

$$\theta_i = ih_1, \quad \gamma_j = jh_2, \quad i = \overline{1, N_1}, \quad j = \overline{1, N_2},$$

where $h_1 = 2\pi/N_1$ and $h_2 = \pi/(N_2 - 1)$ are constant grid steps. For $i = \overline{1, N_1}$ and $j = \overline{2, N_2 - 1}$, we approximate system (19) by the following explicit difference scheme, which is a two-layer scheme with respect to time and is symmetric in space:

$$\frac{\bar{\boldsymbol{u}}_{ij}^{n+1} - \bar{\boldsymbol{u}}_{ij}^n}{\tau_n} + \frac{\bar{\boldsymbol{v}}_{i+1,j}^n - \bar{\boldsymbol{v}}_{i-1,j}^n}{2h_1} + \frac{\bar{\boldsymbol{w}}_{i,j+1}^n - \bar{\boldsymbol{w}}_{i,j-1}^n}{2h_2} = \boldsymbol{f}_{ij}^n + \boldsymbol{W}_{ij}^n. \tag{25}$$

Here

$$\boldsymbol{W}_{ij}^n = C_1 \frac{\bar{\boldsymbol{v}}_{i+1,j}^n - 2\bar{\boldsymbol{v}}_{ij}^n + \bar{\boldsymbol{v}}_{i-1,j}^n}{h_1} + C_2 \frac{\bar{\boldsymbol{w}}_{i,j+1}^n - 2\bar{\boldsymbol{w}}_{ij}^n + \bar{\boldsymbol{w}}_{i,j-1}^n}{h_2}$$

is the artificial viscosity with the coefficients C_1 and C_2 chosen on the basis of test calculations.

Abbreviated notation is introduced for all grid functions in Eq. (25) and below:

$$\bar{\boldsymbol{g}}_{ij}^n = \bar{\boldsymbol{g}}(\alpha_n, \theta_i, \gamma_j), \qquad \boldsymbol{g}_{ij}^n = \boldsymbol{g}(\alpha_n, \theta_i, \gamma_j), \qquad \alpha_n = \sum_{k=0}^{n-1} \tau_k, \qquad \alpha_0 = 0$$

(τ_k is the variable step in terms of the evolutionary variable α). As the sought solution is periodic with respect to the variable θ, Eq. (25) implies that $\theta_0 = \theta_{N_1}$ and $\theta_{N_1+1} = \theta_1$.

Equations (19) degenerate with respect to the variable θ at the boundary nodes $j = 1$ and $j = N_2$, i.e., in the poles $\gamma = 0$ and $\gamma = \pi$. It is known that the principal curvature radii $r_1(\gamma) = \rho(\gamma)$ (12) and $r_2(\gamma) = r(\gamma)/\sin\gamma$, where $r(\gamma)$ is determined by Eq. (11), at each pole on a smooth surface of revolution S (13) coincide with each other, i.e., they are equal to $\rho(0)$ and $\rho(\pi)$, respectively. An analysis of Eqs. (19) shows that system (19) can be written in the following form for $\gamma \to 0$ and $\gamma \to \pi$:

$$\frac{\partial}{\partial\alpha}\,(H_\gamma^2 \boldsymbol{u}) = \boldsymbol{f}. \tag{26}$$

In Eq. (26), we have

$$\boldsymbol{f} = H_\gamma(\sigma_{\theta\theta} + \sigma_{\gamma\gamma}, -\sigma_{\alpha\theta}, -\sigma_{\alpha\gamma})^{\rm t}, \qquad H_\gamma = \rho(\gamma) + \alpha, \qquad \gamma = 0, \pi.$$

For approximate calculations of the grid functions

$$\boldsymbol{g}^n_{i1} = \tilde{\boldsymbol{g}}^n_1, \qquad \boldsymbol{g}^n_{iN_2} = \tilde{\boldsymbol{g}}^n_{N_2}, \qquad i = \overline{1, N_1}$$

at the boundary nodes $j = 1$ and $j = N_2$, we can use the following difference scheme in approximating system (26):

$$\frac{(H^2_\gamma \tilde{\boldsymbol{u}})^{n+1}_m - (H^2_\gamma \tilde{\boldsymbol{u}})^n_m}{\tau_n} = \tilde{\boldsymbol{f}}^n_m, \qquad m = 1, N_2. \tag{27}$$

After finding the quantities $\sigma_{\alpha\alpha}$, $\sigma_{\alpha\theta}$, and $\sigma_{\alpha\gamma}$ at the $(n+1)$th layer from the difference equations (25) and (27) with allowance for Eqs. (18), (20), and (21), the basic grid functions p, n_α, n_θ, and n_γ at each node (i, j) of this layer are determined from the system of algebraic equations (22) and (24). Solving this system under the assumption that $d \equiv \sigma^2_{\alpha\theta} + \sigma^2_{\alpha\gamma} < 1/4$ in the course of calculations, we obtain

$$n_\alpha = \sqrt{(1 + \sqrt{1 - 4d}\,)/2}, \qquad p = \sigma_{\alpha\alpha} - n^2_\alpha, \qquad n_\theta = \sigma_{\alpha\theta}/n_\alpha, \qquad n_\gamma = \sigma_{\alpha\gamma}/n_\alpha.$$

The quantities $\sigma_{\theta\theta}$, $\sigma_{\gamma\gamma}$, and $\sigma_{\theta\gamma}$ involved in system (19)–(21) are calculated by Eqs. (23).

The step τ_n in terms of the evolutionary variable α is determined from the Courant stability condition [9] by the formula

$$\tau_n = \frac{\varkappa(\alpha_n + \min_j \rho(\gamma_j)) \min{(h_1, h_2)}}{\max_{i,j}{(\lambda^n_{ij}, (\lambda^{-1})^n_{ij})}},$$

where λ and λ^{-1} are the velocities of propagation of small perturbations near the surface $S(\alpha_n)$, $\lambda = \sqrt{(1 + 2q)/(1 - 2q)}$, $q = n_\alpha\sqrt{n^2_\theta + n^2_\gamma}$, and $\varkappa \leqslant 1$ is the factor of safety in the condition of stability.

The difference scheme (25), (27) was tested on a problem that has an analytical solution. We considered a thick-walled sphere with an inner radius $a = 1.5$ and outer radius $b = 1.5$e subjected to internal pressure $\tilde{p}$. The solution of this problem is known:

$$\sigma_{rr} = -\tilde{p} + 4k \ln{(r/a)}, \qquad \sigma_{\theta\theta} = \sigma_{rr} + 2k, \qquad \sigma_{\gamma\gamma} = \sigma_{rr} + 2k. \tag{28}$$

Here $\tilde{p} = 4k \ln{(b/a)}$ is the ultimate pressure, σ_{rr} corresponds to the first principal stress (4), and $\sigma_{\theta\theta}$ and $\sigma_{\gamma\gamma}$ correspond to two other principal stresses (5) with $\varepsilon = -1$.

In the variables p, n_α, n_θ, and n_γ, the analytical solution (28) has the form

$$\sigma_{\alpha\alpha}(\alpha) = p(\alpha) + n^2_\alpha(\alpha), \qquad \sigma_{\theta\theta}(\alpha) = \sigma_{\gamma\gamma}(\alpha) = p(\alpha), \tag{29}$$

where

$$p(\alpha) = p(0) - 2\ln{(1 + \alpha/a)}, \qquad p(0) = \tilde{p}/(2k) - 1,$$

$$n_\alpha(\alpha) = 1, \qquad n_\theta(\alpha) = n_\gamma(\alpha) = 0.$$

The following initial values of the functions p, n_α, n_θ, and n_γ were set in the numerical solution on the inner surface of the sphere at $\alpha = 0$:

$$p(0) = 1, \qquad n_\alpha(0) = 1, \qquad n_\theta(0) = n_\gamma(0) = 0.$$

As a result, a solution independent of θ and γ was obtained.

Figure 2 shows the dependences $\sigma_{\alpha\alpha}(\alpha)$ for the exact solution (29) (solid curve) and numerical solution (points). These dependences are in good agreement, which testifies that the scheme proposed is fairly effective.

As an example, we solved the problem of propagation of plastic state zones from the boundary of an ellipsoidal cavity formed by rotating around the z axis of a convex curve L with a support function $F(\gamma) = \sqrt{a^2 \sin^2\gamma + b^2 \cos^2\gamma}$, where a and b are the ellipse semi-axes.

The following values of parameters were used in the numerical solution: $a = 2$, $b = 1$, $N_1 = 200$, and $N_2 = 101$ (N_1 and N_2 are the numbers of nodes in terms of the variables θ and γ, respectively). At $\alpha = 0$, the initial values $p(0) = 1$, $n_\alpha(0) = 1$, and $n_\theta(0) = n_\gamma(0) = 0$ were set on the cavity surface. By virtue of the axial symmetry of the problem, the resultant numerical solution is independent of the angle θ.

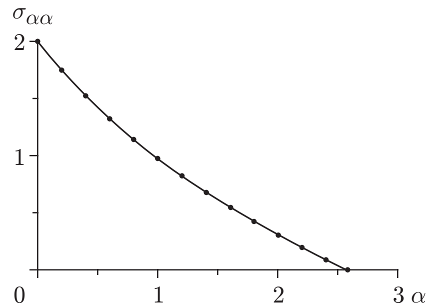

Fig. 2. Distribution of stress $\sigma_{\alpha\alpha}$ in a thick-walled sphere: the solid curve and the points refer to the analytical and numerical solutions, respectively.

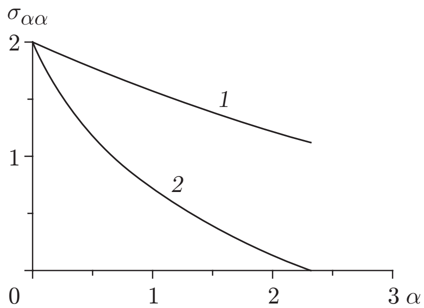

Fig. 3

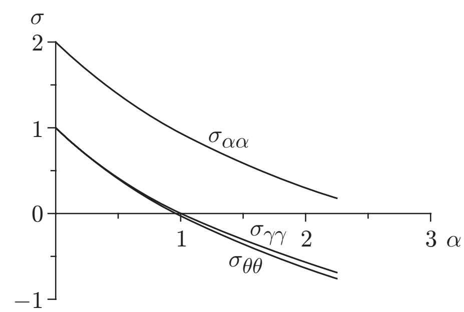

Fig. 4

Fig. 3. Stress $\sigma_{\alpha\alpha}$ versus the coordinate α for $\gamma = 0$ (1) and $\pi/2$ (2).

Fig. 4. Stresses $\sigma_{\alpha\alpha}$, $\sigma_{\theta\theta}$, and $\sigma_{\gamma\gamma}$ versus the coordinate α ($\gamma = \pi/4$).

Figure 3 shows the stress $\sigma_{\alpha\alpha}$ as a function of the coordinate α for $\gamma = 0$ and $\pi/2$. For $\gamma = 0$ and $\pi/2$, the component n_α of the unit eigenvector $\boldsymbol{n}$ (24) equals 1 for all values of α; therefore, according to Eqs. (22) and (23), we have $\sigma_{\theta\theta} = \sigma_{\gamma\gamma} = \sigma_{\alpha\alpha} - 1$.

Figure 4 shows the stresses $\sigma_{\alpha\alpha} = p + n_\alpha^2$, $\sigma_{\theta\theta} = p$, and $\sigma_{\gamma\gamma} = p + n_\gamma^2$ as functions of α for $\gamma = \pi/4$.

Note that a non-axisymmetric solution is obtained for initial conditions depending on θ.

If the stress tensor is known, the velocity vector can be determined from the conditions of incompressibility and isotropy [1] by a similar scheme.

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REFERENCES

1. A. Yu. Ishlinskii and D. D. Ivlev, *Mathematical Theory of Plasticity* [in Russian], Fizmatlit, Moscow (2001).
2. D. D. Ivlev, A. Yu. Ishlinskii, and R. I. Nepershin, "Characteristic relations for stresses and displacement velocities in a spatial problem of an ideally plastic solid under conditions of complete plasticity," *Dokl. Ross. Akad. Nauk*, **381**, No. 5, 616–622 (2001).
3. B. D. Annin, V. O. Bytev, and S. I. Senashov, *Group Properties of Elasticity and Plasticity Equations* [in Russian], Nauka, Novosibirsk (1985).
4. I. S. Gromeka, *Collected Papers* [in Russian], Izd. Akad. Nauk SSSR, Moscow (1952).
5. Yu. N. Radaev, "On the theory of three-dimensional equations of the mathematical theory of plasticity," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 5, 102–120 (2003).

6. Yu. N. Radaev, *A Three-Dimensional Problem of the Mathematical Theory of Plasticity* [in Russian], Izd. Samar. Univ., Samara (2006).
7. A. F. Voevodin and S. M. Shugrin, *Methods for Solving One-Dimensional Evolutionary Systems* [in Russian], Nauka, Novosibirsk (1993).
8. V. V. Ostapenko, *Hyperbolic Systems of Conservation Laws and Their Application to the Shallow Water Theory* (course of lectures) [in Russian], Izd. Novosib. Gos. Univ., Novosibirsk (2004).
9. A. G. Kulikovskii, N. V. Pogorelov, and A. Yu. Semenov, *Mathematical Problems of the Numerical Solution of Hyperbolic Systems of Equations* [in Russian], Fizmatlit, Moscow (2001).